

# $\mathcal{N}$ -fold Supersymmetry in Quantum Mechanical Matrix Models

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## Abstract

We formulate  $\mathcal{N}$ -fold supersymmetry in quantum mechanical matrix models. As an example, we construct general two-by-two Hermitian matrix 2-fold supersymmetric quantum mechanical systems. We find that there are two inequivalent such systems, both of which are characterized by two arbitrary scalar functions, and one of which does not reduce to the scalar system. The obtained systems are all weakly quasi-solvable.

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## I. INTRODUCTION

Recently, supersymmetry (SUSY) and shape invariance in quantum mechanical matrix models have attracted much attention in the literature, e.g., Refs. [1–14] in various physical contexts. To avoid confusion, we note that we here mean SUSY between two matrix Schrödinger operators intertwined by a matrix linear differential operator but *not* SUSY between two scalar Schrödinger operators in a matrix superHamiltonian intertwined by a scalar linear differential operator.

On the other hand, the framework of  $\mathcal{N}$ -fold SUSY [15–17] has been shown to be quite fruitful among several generalizations of ordinary SUSY especially since the establishment of its equivalence with weak quasi-solvability in Ref. [16], for a review see Ref. [18]. Due to the facts that the  $\mathcal{N} = 1$  case corresponds to ordinary SUSY and that shape invariance automatically implies weak quasi-solvability,  $\mathcal{N}$ -fold SUSY contains both ordinary SUSY and shape invariance as its particular cases.

Hence, it is quite natural to ask whether a formulation of  $\mathcal{N}$ -fold SUSY is possible for matrix systems. To the best of our knowledge, there was so far only one such an attempt which corresponds to the  $\mathcal{N} = 2$  case for  $2 \times 2$  matrix models [4]. However, the analysis there was quite restrictive, was devoted mostly to the particular cases where the systems were available by two successive SUSY transformations, and resorted to the involved assumptions and ansatz. As a consequence, its formulation does not have a form which is suitable for discussing general aspects, especially those which were established and appreciated later after that work.

In this article, we formulate for the first time  $\mathcal{N}$ -fold SUSY for a system composed of matrix Schrödinger operators for all positive integral  $\mathcal{N}$  in such a general fashion that the recent crucial developments in the field are fully incorporated in the formalism. To see such a system actually exists, we construct as an illustration general  $2 \times 2$  Hermitian matrix 2-fold SUSY systems without any assumption or ansatz. We find that there are two inequivalent systems, both of which are characterized by two arbitrary real scalar functions. Intriguingly, one of the systems does not admit reduction to the most general 2-fold SUSY scalar system.

We organize this article as follows. In the next section, we generically define  $\mathcal{N}$ -fold SUSY in quantum mechanical matrix models. Then, we investigate in detail  $2 \times 2$  Hermitian matrix systems for  $\mathcal{N} = 2$  in Section III. We explicitly solve all the conditions for 2-fold SUSY to obtain general form of the latter systems. In the last section, we refer to several future issues to be followed after this work.

## II. GENERAL SETTING

A quantum mechanical system we shall consider here is a pair of  $n \times n$  matrix Schrödinger operators

$$H^\pm = -\frac{1}{2}I_n \frac{d^2}{dq^2} + V^\pm(q), \quad (1)$$

where  $I_n$  is an  $n \times n$  unit matrix and each  $V^\pm(q)$  is an  $n \times n$  matrix-valued complex function. Let us introduce a pair of  $n \times n$  matrix linear differential operators of order  $\mathcal{N}$

$$P_{\mathcal{N}}^- = I_n \frac{d^{\mathcal{N}}}{dq^{\mathcal{N}}} + \sum_{k=0}^{\mathcal{N}-1} w_k(q) \frac{d^k}{dq^k}, \quad (2a)$$

$$P_{\mathcal{N}}^+ = (-1)^{\mathcal{N}} I_n \frac{d^{\mathcal{N}}}{dq^{\mathcal{N}}} + \sum_{k=0}^{\mathcal{N}-1} (-1)^k \frac{d^k}{dq^k} w_k(q), \quad (2b)$$

where  $w_k(q)$  ( $k = 0, \dots, \mathcal{N} - 1$ ) are  $n \times n$  matrix-valued complex functions. Then, the system (1) is said to be  $\mathcal{N}$ -fold supersymmetric with respect to (2) if the following relations are all satisfied:

$$P_{\mathcal{N}}^\mp H^\mp - H^\pm P_{\mathcal{N}}^\mp = 0, \quad (3a)$$

$$P_{\mathcal{N}}^\mp P_{\mathcal{N}}^\pm = 2^{\mathcal{N}} \left[ (H^\pm + C_0)^{\mathcal{N}} + \sum_{k=1}^{\mathcal{N}-1} C_k (H^\pm + C_0)^{\mathcal{N}-k-1} \right], \quad (3b)$$

where  $C_k$  ( $k = 0, \dots, \mathcal{N} - 1$ ) are  $n \times n$  constant matrices. It is evident that in the case of  $n = 1$  the above definition of  $\mathcal{N}$ -fold SUSY reduces to the ordinary one for a pair of scalar Schrödinger operators. As in the scalar case, the first relation (3a) immediately implies almost isospectrality of  $H^\pm$  and *weak quasi-solvability*  $H^\pm \ker P_{\mathcal{N}}^\pm \subset \ker P_{\mathcal{N}}^\pm$ . We note that in contrast to the formulation in Ref. [4] where its focus was on hidden symmetry operators characterized as the deviation from the second algebraic relation (3b) we have treated it as a part of the definition of  $\mathcal{N}$ -fold SUSY. See also the discussion in Section IV, item 1 for its relevance.

### III. TWO-BY-TWO HERMITIAN 2-FOLD SUPERSYMMETRY

As an example, let us construct the most general  $2 \times 2$  Hermitian matrix 2-fold SUSY systems which are defined in the Hilbert space  $L^2$  of two-component functions equipped with the inner product defined by

$$(\phi, \psi) = \int_S dq \phi^\dagger(q) \psi(q), \quad \phi, \psi \in L^2(S), \quad S \subset \mathbb{R}, \quad (4)$$

where the Hermitian conjugate  $\dagger$  is as usual the combination of complex conjugate and transposition. The most general form of a pair of  $2 \times 2$  Hermitian matrix Schrödinger operators is given by

$$H^\pm = -\frac{1}{2} I_2 \frac{d^2}{dq^2} + \sum_{\mu=0}^3 V_\mu^\pm(q) \sigma_\mu, \quad (5)$$

where  $\sigma_0 = I_2$  is a  $2 \times 2$  unit matrix,  $\sigma_i$  ( $i = 1, 2, 3$ ) are the Pauli matrices, and  $V_\mu(q)$  ( $\mu = 0, \dots, 3$ ) are all real scalar functions. Components of  $2 \times 2$  matrix 2-fold supercharges

have the following form

$$P_2^- = I_2 \frac{d^2}{dq^2} + \left( \sum_{\mu=0}^3 w_{1\mu}(q) \sigma_\mu \right) \frac{d}{dq} + \sum_{\mu=0}^3 w_{0\mu}(q) \sigma_\mu, \quad (6a)$$

$$P_2^+ = I_2 \frac{d^2}{dq^2} - \frac{d}{dq} \left( \sum_{\mu=0}^3 w_{1\mu}(q) \sigma_\mu \right) + \sum_{\mu=0}^3 w_{0\mu}(q) \sigma_\mu, \quad (6b)$$

where  $w_{1\mu}(q)$  and  $w_{0\mu}(q)$  ( $\mu = 0, \dots, 3$ ) are all real scalar functions. To investigate the  $\mathcal{N}$ -fold SUSY condition (3) for the  $\mathcal{N} = 2$  case under consideration, we first note that  $P_2^- H^- - H^+ P_2^- = 0$  implies  $P_2^+ H^+ - H^- P_2^+ = 0$  and vice versa since they are Hermitian conjugate with each other with respect to the inner product (4), thanks to the choices (5) and (6). Hence, it is sufficient to study only the former. A direct calculation shows that the intertwining relation  $P_2^- H^- - H^+ P_2^- = 0$  holds if and only if the following set of conditions is satisfied:

$$V_\mu^+ - V_\mu^- = w'_{1\mu}, \quad (7)$$

$$w''_{10} + 2w'_{00} + 4V_0^{-'} - 2 \sum_{\mu=0}^3 w_{1\mu} (V_\mu^+ - V_\mu^-) = 0, \quad (8)$$

$$w''_{1i} + 2w'_{0i} + 4V_i^{-'} - 2w_{1i} (V_0^+ - V_0^-) - 2w_{10} (V_i^+ - V_i^-) = 0, \quad (9)$$

$$\sum_{j,k=1}^3 \epsilon_{ijk} w_{1j} (V_k^+ + V_k^-) = 0, \quad (10)$$

$$w''_{00} + 2V_0^{-''} + 2 \sum_{\mu=0}^3 [w_{1\mu} V_\mu^{-'} - w_{0\mu} (V_\mu^+ - V_\mu^-)] = 0, \quad (11)$$

$$w''_{0i} + 2V_i^{-''} + 2w_{1i} V_0^{-'} + 2w_{10} V_i^{-'} - 2w_{0i} (V_0^+ - V_0^-) - 2w_{00} (V_i^+ - V_i^-) = 0, \quad (12)$$

$$\sum_{j,k=1}^3 \epsilon_{ijk} [w_{1j} V_k^{-'} + w_{0j} (V_k^+ + V_k^-)] = 0. \quad (13)$$

On the other hand, we find that the 2-fold superalgebra  $P_2^\mp P_2^\pm = 4[(H^\pm + C_0)^2 + C_1]$  with Hermitian constant matrices  $C_k = \sum_{\mu=0}^3 C_{k\mu} \sigma_\mu$  ( $C_{k\mu} \in \mathbb{R}$ ,  $k = 0, 1$ ) holds for the upper sign

if and only if

$$4V_0^+ = 3w'_{10} - 2w_{00} + \sum_{\mu=0}^3 (w_{1\mu})^2 - 4C_{00}, \quad (14)$$

$$4V_i^+ = 3w'_{1i} - 2w_{0i} + 2w_{10}w_{1i} - 4C_{0i}, \quad (15)$$

$$\sum_{j,k=1}^3 \epsilon_{ijk} w_{1j} (w'_{1k} - w_{0k}) = 0, \quad (16)$$

$$2V_0^{+''} - 4 \sum_{\mu=0}^3 (V_\mu^+ + C_{0\mu})^2 - 4C_{10} = w'''_{10} - w''_{00} + \sum_{\mu=0}^3 [w_{1\mu} w''_{1\mu} + w'_{1\mu} w_{0\mu} - w_{1\mu} w'_{0\mu} - (w_{0\mu})^2], \quad (17)$$

$$2V_i^{+''} - 8(V_0^+ + C_{00})(V_i^+ + C_{0i}) - 4C_{1i} = w'''_{1i} - w''_{0i} + w''_{10} w_{1i} + w_{10} w''_{1i} - w'_{00} w_{1i} + w_{00} w'_{1i} + w'_{10} w_{0i} - w_{10} w'_{0i} - 2w_{00} w_{0i}, \quad (18)$$

$$\sum_{j,k=1}^3 \epsilon_{ijk} (w''_{1j} w_{1k} + w'_{1j} w_{0k} + w_{1j} w'_{0k}) = 0, \quad (19)$$

and for the lower sign if and only if

$$4V_0^- = -w'_{10} - 2w_{00} + \sum_{\mu=0}^3 (w_{1\mu})^2 - 4C_{00}, \quad (20)$$

$$4V_i^- = -w'_{1i} - 2w_{0i} + 2w_{10}w_{1i} - 4C_{0i}, \quad (21)$$

$$\sum_{j,k=1}^3 \epsilon_{ijk} w_{1j} w_{0k} = 0, \quad (22)$$

$$2V_0^{-''} - 4 \sum_{\mu=0}^3 (V_\mu^- + C_{0\mu})^2 - 4C_{10} = -w''_{00} + \sum_{\mu=0}^3 [w'_{1\mu} w_{0\mu} + w_{1\mu} w'_{0\mu} - (w_{0\mu})^2], \quad (23)$$

$$2V_i^{-''} - 8(V_0^- + C_{00})(V_i^- + C_{0i}) - 4C_{1i} = -w''_{0i} + w'_{00} w_{1i} + w_{00} w'_{1i} + w'_{10} w_{0i} + w_{10} w'_{0i} - 2w_{00} w_{0i}, \quad (24)$$

$$\sum_{j,k=1}^3 \epsilon_{ijk} (w'_{1j} w_{0k} + w_{1j} w'_{0k}) = 0. \quad (25)$$

The formulas (14), (15), (20), and (21) determine the form of the potentials  $V_\mu^\pm$  and are compatible with the conditions (7), (8), and (9). Then, the conditions (10), (16), and (22) are identical with

$$\sum_{j,k=1}^3 \epsilon_{ijk} w_{1j} w_{0k} = \sum_{j,k=1}^3 \epsilon_{ijk} w_{1j} w'_{1k} = \sum_{j,k=1}^3 \epsilon_{ijk} w_{1j} C_{0k} = 0. \quad (26)$$

The most general solutions to the latter set of conditions are given by

$$w_{1i} = C_{0i} v_1, \quad w_{0i} = C_{0i} v_0, \quad (27)$$

where  $v_1$  and  $v_0$  are at present arbitrary scalar functions. Substituting (27) into (14), (15), (20), and (21), we obtain

$$4V_0^+ = 3w'_{10} - 2w_{00} + (w_{10})^2 + C^2(v_1)^2 - 4C_{00}, \quad (28a)$$

$$4V_i^+ = C_{0i}(3v'_1 - 2v_0 + 2w_{10}v_1 - 4), \quad (28b)$$

$$4V_0^- = -w'_{10} - 2w_{00} + (w_{10})^2 + C^2(v_1)^2 - 4C_{00}, \quad (28c)$$

$$4V_i^- = C_{0i}(-v'_1 - 2v_0 + 2w_{10}v_1 - 4), \quad (28d)$$

where  $C^2 = \sum_{i=1}^3 (C_{0i})^2$ . The solutions (27) automatically satisfy (13), (19), and (25). With the substitution of (27) and (28) into the remaining conditions, (11) and (12) are respectively identical with

$$\begin{aligned} & w'''_{10} - w_{10}w''_{10} - 2(w'_{10})^2 + 4w'_{10}w_{00} + 2w_{10}w'_{00} - 2(w_{10})^2w'_{10} \\ & - C^2 [v_1v''_1 + 2(v'_1)^2 - 4v'_1v_0 - 2v_1v'_0 + 2w'_{10}(v_1)^2 + 4w_{10}v_1v'_1] = 0, \end{aligned} \quad (29)$$

$$\begin{aligned} & v'''_1 - w''_{10}v_1 - 4w'_{10}v'_1 - w_{10}v''_1 + 4w'_{10}v_0 + 2w_{10}v'_0 + 2w'_{00}v_1 + 4w_{00}v'_1 \\ & - 4w_{10}w'_{10}v_1 - 2(w_{10})^2v'_1 - 2C^2(v_1)^2v'_1 = 0, \end{aligned} \quad (30)$$

(17) and (18) are respectively with

$$\begin{aligned} & 2w'''_{10} - 5(w'_{10})^2 + 8w'_{10}w_{00} + 4w_{10}w'_{00} - 6(w_{10})^2w'_{10} + 4(w_{10})^2w_{00} - (w_{10})^4 \\ & - C^2 [5(v'_1)^2 - 8v'_1v_0 - 4v_1v'_0 + 6w'_{10}(v_1)^2 + 12w_{10}v_1v'_1 - 8w_{10}v_1v_0 \\ & - 4w_{00}(v_1)^2 + 6(w_{10})^2(v_1)^2] - C^4(v_1)^4 - 16C_{10} = 0, \end{aligned} \quad (31)$$

$$\begin{aligned} & C_{0i} [v'''_1 - 5w'_{10}v'_1 + 4w'_{10}v_0 + 2w_{10}v'_0 + 2w'_{00}v_1 + 4w_{00}v'_1 - 6w_{10}w'_{10}v_1 - 3(w_{10})^2v'_1 \\ & + 2(w_{10})^2v_0 + 4w_{10}w_{00}v_1 - 2(w_{10})^3v_1 - C^2(v_1)^2(3v'_1 - 2v_0 + 2w_{10}v_1)] - 8C_{1i} = 0, \end{aligned} \quad (32)$$

and (23) and (24) are respectively with

$$\begin{aligned} & 2w'''_{10} - 4w_{10}w''_{10} - 3(w'_{10})^2 + 8w'_{10}w_{00} + 4w_{10}w'_{00} - 2(w_{10})^2w'_{10} - 4(w_{10})^2w_{00} \\ & + (w_{10})^4 - C^2 [4v_1v''_1 + 3(v'_1)^2 - 8v'_1v_0 - 4v_1v'_0 + 2w'_{10}(v_1)^2 + 4w_{10}v_1v'_1 \\ & + 8w_{10}v_1v_0 + 4w_{00}(v_1)^2 - 6(w_{10})^2(v_1)^2] + C^4(v_1)^4 + 16C_{10} = 0, \end{aligned} \quad (33)$$

$$\begin{aligned} & C_{0i} [v'''_1 - 2w''_{10}v_1 - 3w'_{10}v'_1 - 2w_{10}v''_1 + 4w'_{10}v_0 + 2w_{10}v'_0 + 2w'_{00}v_1 + 4w_{00}v'_1 \\ & - 2w_{10}w'_{10}v_1 - (w_{10})^2v'_1 - 2(w_{10})^2v_0 - 4w_{10}w_{00}v_1 + 2(w_{10})^3v_1 \\ & + C^2(v_1)^2(-v'_1 - 2v_0 + 2w_{10}v_1)] + 8C_{1i} = 0. \end{aligned} \quad (34)$$

It is evident that the equations (32) and (34) have the trivial solutions  $C_{1i} = C_{0i} = 0$ . On the other hand, for non-trivial solutions the combination  $C_{1i}/C_{0i} := \tilde{C}$  ( $i = 1, 2, 3$ ) should not depend on the index  $i$ .

Let us first consider the set of conditions (29) and (30). We find that the two combinations  $w_{10} \times (29) + C^2 v_1 \times (30)$  and  $v_1 \times (29) + w_{10} \times (30)$  are total differentials and thus are integrated respectively as

$$\begin{aligned} & 2w_{10}w''_{10} - (w'_{10})^2 - 2(w_{10})^2w'_{10} + 4(w_{10})^2w_{00} - (w_{10})^4 \\ & + C^2 [2v_1v''_1 - (v'_1)^2 - 2w'_{10}(v_1)^2 - 4w_{10}v_1v'_1 + 8w_{10}v_1v_0 \\ & + 4w_{00}(v_1)^2 - 6(w_{10})^2(v_1)^2] - C^4(v_1)^4 = D_1, \end{aligned} \quad (35)$$

and

$$w_{10}''v_1 - w_{10}'v_1' + w_{10}v_1'' - 2w_{10}w_{10}'v_1 - (w_{10})^2v_1' + 2(w_{10})^2v_0 + 4w_{10}w_{00}v_1 - 2(w_{10})^3v_1 - C^2(v_1)^2(v_1' - 2v_0 + 2w_{10}v_1) = D_2, \quad (36)$$

where  $D_1$  and  $D_2$  are integral constants. It is easily checked that (35) and (36) are compatible with all the remaining conditions (31)–(34) if and only if

$$D_1 = 16C_{10}, \quad D_2 = 8C_{1i}/C_{0i} = 8\tilde{C}. \quad (37)$$

Hence, the only remaining problem is to analyze (35) and (36). They can be regarded as simultaneous linear equations for  $w_{00}$  and  $v_0$ . For the non-degenerate case  $v_1 \neq C^{-1}w_{10}$ , they are uniquely solved as

$$\begin{aligned} 4[(w_{10})^2 - C^2(v_1)^2]^2 w_{00} &= (w_{10})^2[-2w_{10}w_{10}'' + (w_{10}')^2 + 2(w_{10})^2w_{10}' + (w_{10})^4 + 16C_{10}] \\ &+ C^2[2w_{10}w_{10}''(v_1)^2 + (w_{10}')^2(v_1)^2 - 4w_{10}w_{10}'v_1v_1' + 2(w_{10})^2v_1v_1'' + (w_{10})^2(v_1')^2 \\ &- 4(w_{10})^2w_{10}'(v_1)^2 - (w_{10})^4(v_1)^2 - 32\tilde{C}w_{10}v_1 + 16C_{10}(v_1)^2] \\ &- C^4(v_1)^2[2v_1v_1'' - (v_1')^2 - 2w_{10}'(v_1)^2 + (w_{10})^2(v_1)^2] + C^6(v_1)^6, \end{aligned} \quad (38)$$

and

$$\begin{aligned} 2[(w_{10})^2 - C^2(v_1)^2]^2 v_0 &= w_{10}[w_{10}w_{10}''v_1 - (w_{10}')^2v_1 + w_{10}w_{10}'v_1' - (w_{10})^2v_1'' + (w_{10})^3v_1' \\ &+ (w_{10})^4v_1 + 8\tilde{C}w_{10} - 16C_{10}v_1] + C^2v_1[-w_{10}''(v_1)^2 + w_{10}'v_1v_1' + 2w_{10}v_1v_1'' - w_{10}(v_1')^2 \\ &- 2(w_{10})^2v_1v_1' - 2(w_{10})^3(v_1)^2 + 8\tilde{C}v_1] + C^4(v_1)^4(v_1' + w_{10}v_1). \end{aligned} \quad (39)$$

Finally, the general form of a  $2 \times 2$  Hermitian matrix 2-fold SUSY system for the non-degenerate case is given by

$$\begin{aligned} H^+ &= -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{4}[3w_{10}' - 2w_{00} + (w_{10})^2 + C^2(v_1)^2] \\ &+ \frac{1}{4}(3v_1' - 2v_0 + 2w_{10}v_1) \sum_{i=1}^3 C_{0i}\sigma_i - C_0, \end{aligned} \quad (40)$$

$$\begin{aligned} H^- &= -\frac{1}{2}\frac{d^2}{dq^2} + \frac{1}{4}[-w_{10}' - 2w_{00} + (w_{10})^2 + C^2(v_1)^2] \\ &+ \frac{1}{4}(-v_1' - 2v_0 + 2w_{10}v_1) \sum_{i=1}^3 C_{0i}\sigma_i - C_0, \end{aligned} \quad (41)$$

$$P_2^- = \frac{d^2}{dq^2} + \left( w_{10} + v_1 \sum_{i=1}^3 C_{0i}\sigma_i \right) \frac{d}{dq} + w_{00} + v_0 \sum_{i=1}^3 C_{0i}\sigma_i. \quad (42)$$

The two functions  $w_{00}$  and  $v_0$  in the above can be eliminated by using (38) and (39). Hence, the system can be expressed solely in terms of the two functions  $w_{10}$  and  $v_1$ .

For the degenerate case  $v_1 = C^{-1}w_{10}$ , the two equations (35) and (36) are not independent and are equivalent with the following single equation

$$4(w_{10})^2(w_{00} + Cv_0) = -2w_{10}w_{10}'' + (w_{10}')^2 + 4(w_{10})^2w_{10}' + 4(w_{10})^4 + 8C_{10}, \quad (43)$$

with  $C_{10} = \tilde{C}C$ . Hence, we can again eliminate two of the four functions, e.g.,  $v_1$  and  $v_0$ . The general form of a  $2 \times 2$  Hermitian matrix 2-fold SUSY system for the degenerate case is given by

$$H^+ = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{1}{4} [3w'_{10} - 2w_{00} + 2(w_{10})^2] + \frac{1}{4C} \left[ w'_{10} + 2w_{00} + \frac{w''_{10}}{w_{10}} - \frac{(w'_{10})^2}{2(w_{10})^2} - \frac{4\tilde{C}C}{(w_{10})^2} \right] \sum_{i=1}^3 C_{0i}\sigma_i - C_0, \quad (44)$$

$$H^- = -\frac{1}{2} \frac{d^2}{dq^2} + \frac{1}{4} [-w'_{10} - 2w_{00} + 2(w_{10})^2] + \frac{1}{4C} \left[ -3w'_{10} + 2w_{00} + \frac{w''_{10}}{w_{10}} - \frac{(w'_{10})^2}{2(w_{10})^2} - \frac{4\tilde{C}C}{(w_{10})^2} \right] \sum_{i=1}^3 C_{0i}\sigma_i - C_0, \quad (45)$$

$$P_2^- = \frac{d^2}{dq^2} + w_{10} \left( 1 + C^{-1} \sum_{i=1}^3 C_{0i}\sigma_i \right) \frac{d}{dq} + w_{00} + \frac{1}{C} \left[ w'_{10} - w_{00} + (w_{10})^2 - \frac{w''_{10}}{2w_{10}} + \frac{(w'_{10})^2}{4(w_{10})^2} + \frac{2\tilde{C}C}{(w_{10})^2} \right] \sum_{i=1}^3 C_{0i}\sigma_i. \quad (46)$$

It is interesting to note that in the limit  $C_{0i} \rightarrow 0$  ( $i = 1, 2, 3$ ), the non-degenerate system (40)–(42) reduces to the most general 2-fold SUSY scalar system [16, 19, 20] while the degenerate system (44)–(46) does *not*.

#### IV. DISCUSSION AND SUMMARY

In this article, we have for the first time formulated generically  $\mathcal{N}$ -fold SUSY in quantum mechanical matrix models and constructed the general  $2 \times 2$  Hermitian matrix 2-fold SUSY systems without recourse to any assumption or ansatz. In addition to the detailed studies for larger  $n \times n$  matrices ( $n > 2$ ) and  $\mathcal{N} > 2$  cases, there are many future issues to be followed after this work as the following:

1. First of all, it is important to clarify general aspects of  $\mathcal{N}$ -fold SUSY in matrix systems, as were done in [16, 17] for scalar systems. In the scalar case, there are two significant features, namely, the equivalence between  $\mathcal{N}$ -fold SUSY and weak quasi-solvability and the equivalence between the conditions (3a) and (3b). In the case of  $2 \times 2$  Hermitian matrix systems, however, it does not seem that the conditions (10) and (13) coming from the former are equivalent with the conditions (16), (19), (22), and (25) coming from the latter although the other conditions are certainly equivalent. That is exactly the reason why we considered the both to derive the formula (27). We expect that the general approach [21] for the scalar case recently proposed by us would be also efficient for the matrix case.
2. In the scalar case, the systematic algorithm for constructing an  $\mathcal{N}$ -fold SUSY system [22] based on quasi-solvability has shown to be quite effective. Hence, its generalization to the matrix case is desirable. It would also enable us to connect directly the possible types of matrix  $\mathcal{N}$ -fold SUSY systems with the possible linear spaces of multi-component functions preserved by a second-order matrix linear differential operator. For example, it is



interesting to see the connection with the quasi-solvable matrix operators constructed from the generators of  $\mathfrak{sl}(2)$  in Ref. [23].

3. Shape invariance is a well-known sufficient condition for *solvability* [24]. It means in particular that it always implies  $\mathcal{N}$ -fold SUSY in the scalar case. In fact, some shape-invariant scalar potentials were systematically constructed as particular cases of  $\mathcal{N}$ -fold SUSY with intermediate Hamiltonians [25, 26]. Recently, several shape-invariant matrix potentials were constructed in Refs. [2, 6, 12–14], and we expect that our formulation of  $\mathcal{N}$ -fold SUSY would be also quite efficient in constructing shape-invariant matrix models.

4. Extension to more general second-order matrix linear differential operators would be possible, e.g., by admitting a non-diagonal second-order operator and by adding a matrix-valued first-order operator. In particular, a quantum mechanical matrix model with matrix-valued position-dependent mass would be an interesting candidate as a natural generalization of  $\mathcal{N}$ -fold SUSY in scalar quantum systems with position-dependent mass [27].

5. In the scalar case, there are several intimate relations between  $\mathcal{N}$ -fold SUSY and  $\mathcal{N}$ th-order paraSUSY [25, 26, 28, 29]. We expect that we can formulate higher-order paraSUSY in quantum mechanical matrix models in a way such that the relations to  $\mathcal{N}$ -fold SUSY in the scalar case remain intact in a matrix case. Extension of higher-order  $\mathcal{N}$ -fold paraSUSY [30] to matrix systems would be also possible.

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